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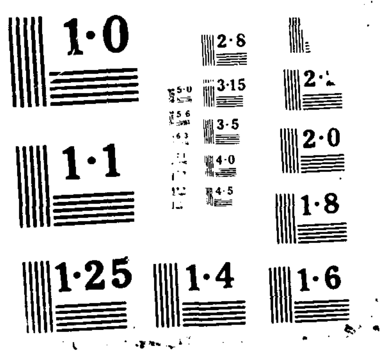
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# A NEW BOUNDARY INTEGRAL EQUATION FORMULATION FOR ELASTODYNAMIC AND ELASTOSTATIC CRACK ANALYSIS

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## Abstract

An elastodynamic conservation integral, the $\tilde{J}_k$ integral, is employed to derive boundary integral equations for crack configurations, in a direct and natural way. These equations immediately have lower order singularities than the ones obtained in the conventional manner by the use of the Betti-Rayleigh reciprocity relation. This is an important advantage for the development of numerical procedures for solving the BIE's, and for an accurate calculation of the strains and stresses at internal points close to the crack faces. For curved cracks of arbitrary shape the BIE's presented here have simple forms, and they do not require integration by parts, as in the conventional formulation. For the dynamic case, the unknown quantities are the crack opening displacements and their derivatives (dislocation densities), while for the static case only the dislocation densities appear in the formulation. For plane cracks the boundary integral equations reduce to the ones obtained by other investigators.

## 1. Introduction

Boundary integral equations, in conjunction with the boundary element method, provide an effective numerical technique for the solution of boundary value problems in solid mechanics. The boundary integral equation method (BIEM) has been successfully applied to a wide range of problems in linear and nonlinear elasticity. Recent developments of the boundary integral equation method have been concerned with applications to elastostatic and elastodynamic crack analysis. The method is attractive for crack analysis, because the semi-analytical nature of the BIEM makes it easy to take into account the singularities at the crack tips.

The conventional BIE formulation, due to Rizzo (1967) and Cruse (1969), is based on the Betti-Rayleigh reciprocity theorem for two independent elastostatic or elastodynamic states. By choosing one of the states as the unknown field and the other as the basic singular solution (the Green's function), a representation integral for the displacement components can be derived. The integral, which is over the surface of the crack contains the crack opening displacements (the displacement jumps across the crack faces) and derivatives of the Green's function in its integrand. Unfortunately, a direct limiting process on the representation integral for the displacements as the observation point approaches a crack face, gives rise to a degenerate set of BIE's, as shown by Cruse (1975). This has motivated the use of representation integrals for the tractions, and their corresponding boundary integral equations, rather than displacement BIE's. Such traction BIE's are, however, highly singular, and they cannot be solved directly by numerical methods. To circumvent these difficulties several approaches have been proposed, see for example the papers by Cruse (1975,1987), Weaver

(1977), Budiansky and Rice (1979), Schmerr (1982), Sládek and Sládek (1984), Nishimura and Kobayashi (1987), Zhang and Achenbach (1988) and Budreck and Achenbach (1988). Most of these studies first reduce the higher order singularities to integrable ones, and then solve the modified BIE's numerically. The reduction is achieved by the use of partial integration. The required manipulations are reasonably easy for simple configurations such as 3-D planar or 2-D straight cracks, but they become cumbersome for curved cracks. Furthermore, different forms of the regularized BIE's are obtained through the non-unique integration-by-parts process, though they are equivalent (see Cruse, 1987).

In this paper we present a new BIE formulation for crack analysis. The motivation for this study is the paper by Hu (1987), who proposed a novel way to obtain a new type of BIE, to solve elastostatic boundary value problems. Hu's formulation is based on the conservation integral $J_k$. In the present paper it is shown that Hu's BIE's are especially suited for solving crack problems. For elastodynamic problems, the $J_k$ integral of elastostatics is generalized to time-harmonic elastodynamics, and the result is denoted by $\tilde{J}_k$. Boundary integral equations are then derived from $\tilde{J}_k$ in a direct and natural manner, for arbitrary crack configurations. The BIE's that are obtained are immediately less singular than the ones of the conventional formulation, and they do not require additional manipulation in developing numerical solution procedures. The BIE's presented in this paper have relatively simple forms, and they reduce to those for elastostatics by letting $\omega \to 0$ and by using the appropriate static Kelvin solutions. For planar and straight cracks the results agree with those obtained by other

authors. New BIE's derived from the complementary conservation integral $\tilde{J}_k^c$ are also given, but these equations do not offer advantages over the conventional formulation.

The significance of $J_k$ (or $\tilde{J}_k$) integral as a relevant crack-tip parameter in linear and nonlinear fracture mechanics has been well established (see Moran and Shih, 1987). The present paper presents a novel application of such path independent or conservation integrals in elastodynamic and elastostatic crack analysis.

## 2. Problem Statement and Conventional BIE Formulation

A crack is a surface of displacement discontinuity when external loads are applied to the body. The faces of a mathematical crack are infinitesimally close prior to loading, and they do not interact when loads are applied. This is an acceptable approximation for real cracks whose faces are initially sufficiently separated so that the faces will not touch when the body is disturbed.

In this paper we consider a three-dimensional (curved) crack of arbitrary shape which is contained in an unbounded, homogeneous, isotropic, linearly elastic solid. The geometry is shown in Fig. 1. The solid is subjected to time-harmonic motion, but the term $\exp(-i\omega t)$ has been suppressed throughout the analysis.

The stress equations of motion are given by (see Achenbach, 1973)

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0 , \tag{2.1}$$

where $\sigma_{ij}$ defines the stress components, $u_i$ denotes the displacement components, $\rho$ is the mass density, and $\omega$ is the angular frequency. In Eq.(2.1) body forces are not considered and the summation convention is implied. In the linear theory the strain components are defined as

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad . \tag{2.2}$$

The stress and strain components are related by Hooke's law

$$\sigma_{ij} = C_{ijk\ell}\epsilon_{k\ell} \quad , \tag{2.3}$$

where $C_{ijk\ell}$ are elastic constants which for isotropic materials can be written as

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \quad . \tag{2.4}$$

Here $\lambda$ and $\mu$ are Lamé's elastic constants and $\delta_{ij}$ is the Kronecker delta. The tractions vanish on the faces of the crack, i.e.,

$$f_i = \sigma_{ij}n_j = 0 \quad , \quad \underset{\sim}{x} \in A \quad , \tag{2.5}$$

where $A = A^+ + A^-$. For a scattering problem $A^+$ is the insonified side of the crack and $A^-$ is the shadow side. Also, $n_j$ is the unit normal vector of A.

The total fields generated by the interaction of an incident wave with the crack can be written as

$$u_i = u_i^{in} + u_i^{sc} \quad , \quad \sigma_{ij} = \sigma_{ij}^{in} + \sigma_{ij}^{sc} \quad , \tag{2.6}$$

where $u_i^{in}$ and $\sigma_{ij}^{in}$ represent the incident field in the absence of the crack, and $u_i^{sc}$, $\sigma_{ij}^{sc}$ define the scattered field. Both the total fields and the partial fields satisfy Eqs.(2.1)-(2.3). For a given incident field the scattered field has to satisfy the boundary conditions on the faces of the crack, Eq.(2.5).

Following the procedure proposed by Rizzo (1967) and Cruse (1969), a representation integral for the scattered displacement can be obtained by using the Betti-Rayleigh reciprocal theorem and the fundamental solution due to a unit time-harmonic point force. For a 3-dimensional crack, the representation integral can be written as

$$u_k^{sc}(\underset{\sim}{x}) = \int_{A^+} \sigma_{ijk}^G(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_i(\underset{\sim}{y})n_j dA(\underset{\sim}{y}), \qquad \underset{\sim}{x} \notin A^+ . \tag{2.7}$$

Here $\underset{\sim}{x}$ is the position vector of the observation point, $\sigma_{ijk}^G(\underset{\sim}{x}-\underset{\sim}{y})$ is the stress Green's function (Appendix A), and $\Delta u_i(\underset{\sim}{y})$ defines the displacement jumps (crack opening displacements) across the faces of the crack.

As shown by Cruse (1975) for the static case, Eq.(2.7) will lead to a degenerate BIE formulation as $\underset{\sim}{x} \rightarrow A^+$. A natural remedy for this difficulty

is to use the representation integral for the traction components, which can be obtained by substituting Eq.(2.7) into Hooke's law and by using $f_p^{sc} = \sigma_{pq}^{sc} n_q$. The result is

$$f_p^{sc}(\underset{\sim}{x}) = - C_{pqk\ell} n_q(\underset{\sim}{x}) \int_{A^+} \sigma_{ijk,\ell}^{G}(\underset{\sim}{x}-\underset{\sim}{y}) \Delta u_i(\underset{\sim}{y}) n_j dA(\underset{\sim}{y}), \quad \underset{\sim}{x} \notin A^+ . \tag{2.8}$$

Boundary integral equations can be derived from Eq.(2.8) by letting $\underset{\sim}{x} \to A^+$ and by applying the boundary conditions (2.5). The system of boundary integral equations that is obtained in this manner is, however, hypersingular when the observation point $\underset{\sim}{x}$ and the source point $\underset{\sim}{y}$ coincide, since the terms $\sigma_{ijk,\ell}^{G}(\underset{\sim}{x}-\underset{\sim}{y})$ behave as (Appendix A)

$$\sigma_{ijk,\ell}^{G}(\underset{\sim}{x}-\underset{\sim}{y}) \sim \begin{cases} \dfrac{1}{r^2} , & \text{2-D} , \\ \dfrac{1}{r^3} , & \text{3-D} , \end{cases} \quad \text{as } r \to 0 , \tag{2.9}$$

where $r = |\underset{\sim}{x}-\underset{\sim}{y}|$. These higher order singularities prevent a reliable direct numerical solution of Eq.(2.8).

To overcome these difficulties Budiansky and Rice (1979) used partial integration to reduce the higher order singularities, and to derive a system of BIE's for a flat crack in the plane $x_3 = 0$ (see Fig. 1). Regularization procedures have also been proposed by Sládek and Sládek (1984), by Nishimura and Kobayashi (1987), and by Budreck and Achenbach (1988).

For a two-dimensional crack configuration, analogous formulations have been proposed by Tan (1975), by Schmerr (1982), and by Zhang and Achenbach (1988). The corresponding elastostatic crack analysis using BIE methods has been presented by Cruse (1975) and Weaver (1977). A comprehensive discussion and an extensive list of references has been given by Cruse (1987).

All the studies mentioned above have used partial integration to reduce the higher order singularities (2.9). This procedure is easily implemented for flat or straight cracks, but it becomes quite cumbersome for curved cracks. In this paper, we will present a new BIE formulation which follows very naturally from a path independent integral, and which has lower order singularities than the ones obtained in the conventional BIE formulation.

## 3. The $\bar{J}_k$ Integral and Related BIE's

In elastostatics the $J_k$ integral has the form (Eshelby, 1951, Rice, 1968)

$$J_k = \int_S (W\delta_{jk} - \sigma_{ij}u_{i,k})n_j dS \quad , \tag{3.1}$$

where S is the surface of a body with volume V, $n_j$ is the outward normal vector, and W is the elastic strain energy density

$$W = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \quad . \tag{3.2}$$

The integral $J_k$, which vanishes if there are no body forces and singularities present in V, is usually referred to as a path independent integral or a conservation law. The application of the $J_1$ component as a relevant crack-tip parameter in linear and nonlinear fracture mechanics has been well established (see Moran and Shih, 1987). It can easily be shown that $J_k = 0$, by applying the divergence theorem, by using

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \quad , \tag{3.3}$$

and by employing the equilibrium equations $\sigma_{ij,j} = 0$. The generalization of $J_k$ to time-harmonic elastodynamics, which is denoted by $\tilde{J}_k$, can be written as

$$\tilde{J}_k = \int_S [(W+L)\delta_{jk} - \sigma_{ij}u_{i,k}]n_j dS \quad , \tag{3.4}$$

where L is the kinetic energy density

$$L = \frac{1}{2}\rho\ddot{u}_i u_i = -\frac{1}{2}\rho\omega^2 u_i u_i \quad . \tag{3.5}$$

Here also $\tilde{J}_k = 0$, under the same assumptions as for $J_k$. The proof is again very simple if we apply the divergence theorem, use Eq.(3.3) and employ the

equations of motion (2.1). We note that $\tilde{J}_k = 0$ holds for any time-harmonic elastodynamic state which satisfies Eqs.(2.1)-(2.3).

Now let us consider two independent time-harmonic elastodynamic states for the same body:

$$u_i^{(1)} \ , \quad \epsilon_{ij}^{(1)} \ , \quad \sigma_{ij}^{(1)} \quad , \tag{3.6}$$

$$u_i^{(2)} \ , \quad \epsilon_{ij}^{(2)} \ , \quad \sigma_{ij}^{(2)} \quad . \tag{3.7}$$

These states satisfy the equations of motion (2.1), the strain-displacement equation (2.2) and Hooke's law (2.3). By virtue of linear superposition, the sum of (3.6) and (3.7)

$$u_i = u_i^{(1)} + u_i^{(2)} \ , \ \epsilon_{ij} = \epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)} \ , \quad \sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \ , \tag{3.8}$$

also satisfies Eqs.(2.1)-(2.3). Substitution of (3.8) into (3.4) yields

$$\tilde{J}_k[u_i] = \tilde{J}_k[u_i^{(1)}] + \tilde{J}_k[u_i^{(2)}] + \int_S [(u_{m,n}^{(1)}\sigma_{mn}^{(2)} - \rho\omega^2 u_i^{(1)}u_i^{(2)})\delta_{jk}$$

$$- u_{i,k}^{(1)}\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}u_{i,k}^{(2)}]n_j dS \quad . \tag{3.9}$$

Clearly, the terms $\tilde{J}_k[u_i^{(1)}]$ and $\tilde{J}_k[u_i^{(2)}]$ must vanish because $u_i^{(1)}$ and $u_i^{(2)}$ are two independent elastodynamic states. Since $\tilde{J}_k\,[u_i] = 0$ we therefore conclude that

$$\int_S [(u_{m,n}^{(1)}\sigma_{mn}^{(2)} - \rho\omega^2 u_i^{(1)} u_i^{(2)})\delta_{jk} - u_{i,k}^{(1)}\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)} u_{i,k}^{(2)}]n_j dS = 0 \ . \qquad (3.10)$$

Equation (3.10) is an extension of the two-state conservation integrals proposed by Chen and Shield (1977) for elastostatics ($\omega \to 0$). The first state is now taken to be the unknown field

$$\{u_i^{(1)}, \sigma_{ij}^{(1)}\} = \{u_i^{sc}, \sigma_{ij}^{sc}\} \quad , \qquad (3.11)$$

while the second state is selected as the fundamental solution due to a unit point force

$$\{u_i^{(2)}, \sigma_{ij}^{(2)}\} = \{u_{i\ell}^G a_\ell, \sigma_{ij\ell}^G a_\ell\} \quad , \qquad (3.12)$$

where $u_{i\ell}^G$ and $\sigma_{ij\ell}^G$ are 3-D time-harmonic elastodynamic Green's functions (see Appendix A), and $a_\ell$ indicates the directions of the applied point force. Application of (3.10) to the surfaces S, $S_\delta$ and $S_R$ (Fig.2), and use of the Eqs.(3.11) and (3.12) results in

$$-\int_S I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y})dS(\underset{\sim}{y}) + \int_{S_\delta} I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y})dS(\underset{\sim}{y}) + \int_{S_R} I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y})dS(\underset{\sim}{y}) = 0 , \qquad (3.13)$$

where S is the surface of the scatterer, $S_\delta$ is the surface of a sphere of radius $\delta$, centered at $\underset{\sim}{x}$, and $S_R$ is the surface of a sphere with radius R, centered at $\underset{\sim}{x}$, as shown in Fig. 2. The surface S is assumed to be closed, regular and smooth. The small sphere (radius $\delta$) is selected to exclude the singularities in the Green's functions, and the sphere with radius R must be sufficiently large so that the scatterer S and the sphere $S_\delta$ are included in it. The integrand $I_{\ell k}$ in (3.13) is given by

$$I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y}) = \{[u^{sc}_{m,n}(\underset{\sim}{y})\sigma^G_{mn\ell}(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 u^{sc}_i(\underset{\sim}{y})u^G_{i\ell}(\underset{\sim}{x}-\underset{\sim}{y})]\delta_{jk}$$

$$- u^{sc}_{i,k}(\underset{\sim}{y})\sigma^G_{ij\ell}(\underset{\sim}{x}-\underset{\sim}{y}) - \sigma^{sc}_{ij}(\underset{\sim}{y})u^G_{i\ell,k}(\underset{\sim}{x}-\underset{\sim}{y})\}n_j , \qquad (3.14)$$

in which $\underset{\sim}{x}$ represents the position vector of the observation point, and $\underset{\sim}{y}$ represents the position vector of the source point, respectively.

After elementary calculations the second integral in Eq.(3.13) can be evaluated as

$$\int_{S_\delta} I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y})dS(\underset{\sim}{y}) = - u^{sc}_{\ell,k}(\underset{\sim}{x}) , \quad \text{as } \delta \to 0 . \qquad (3.15)$$

By using the asymptotic expansions of the Green's functions for large $|\underset{\sim}{y}|$ (Appendix B) the last integral in Eq.(3.13) can be rewritten in the following form

$$\int_{S_R} I_{\ell k}(\underset{\sim}{x};\underset{\sim}{y})dS(\underset{\sim}{y}) = -\frac{ik_L}{\lambda+2\mu}\frac{\exp(ik_LR)}{4\pi R}\int_{S_R}\{[\sigma_{ij}^{sc}n_j - i(\lambda+2\mu)k_L u_i^{sc}]n_i\}\cdot$$

$$n_\ell n_k \exp(-ik_L\underset{\sim}{n}\cdot\underset{\sim}{x})dS(\underset{\sim}{y}) -$$

$$-\frac{ik_T}{\mu}\frac{\exp(ik_TR)}{4\pi R}\int_{S_R}\{(\sigma_{ij}^{sc}n_j - i\mu k_T u_i^{sc})\cdot$$

$$(\delta_{i\ell} - n_i n_\ell)n_k\}\exp(-ik_T\underset{\sim}{n}\cdot\underset{\sim}{x})dS(\underset{\sim}{y}). \tag{3.16}$$

Applying the Cauchy-Schwartz inequality

$$\left|\int_{S_R} f(\underset{\sim}{y})dS(\underset{\sim}{y})\right| \le \left(\int_{S_R} dS(\underset{\sim}{y})\right)^{\frac{1}{2}}\left(\int_{S_R}|f(\underset{\sim}{y})|^2 dS(\underset{\sim}{y})\right)^{\frac{1}{2}}$$

$$= 2\sqrt{\pi}\, R\left(\int_{S_R}|f(\underset{\sim}{y})|^2 dS(\underset{\sim}{y})\right)^{\frac{1}{2}}, \tag{3.17}$$

and considering the following elastodynamic radiation conditions (see Tan, 1975, Achenbach, 1982)

$$\lim_{R\to\infty} \int_{S_R} |[\sigma_{ij}^{sc}n_j - i(\lambda+2\mu)k_L u_i^{sc}]n_i|^2 dS(\underline{y}) = 0 \; , \tag{3.18}$$

$$\lim_{R\to\infty} \int_{S_R} |(\sigma_{ij}^{sc}n_j - i\mu k_T u_i^{sc})(\delta_{i\ell} - n_i n_\ell)|^2 dS(\underline{y}) = 0 \; , \tag{3.19}$$

we obtain

$$\int_{S_R} I_{\ell k}(\underline{x};\underline{y}) dS(\underline{y}) = 0 \quad , \quad \text{as } R \to \infty \quad . \tag{3.20}$$

Thus, Eq.(3.13) is reduced to

$$u_{\ell,k}^{sc}(\underline{x}) = - \int_S \{[u_{m,n}^{sc}(\underline{y})\sigma_{mn\ell}^G(\underline{x}-\underline{y}) - \rho\omega^2 u_i^{sc}(\underline{y})u_{i\ell}^G(\underline{x}-\underline{y})]\delta_{jk} -$$

$$- u_{i,k}^{sc}(\underline{y})\sigma_{ij\ell}^G(\underline{x}-\underline{y}) - \sigma_{ij}^{sc}(\underline{y})u_{i\ell,k}^G(\underline{x}-\underline{y})\} n_j dS(\underline{y}) \; ,$$

$$\underline{x} \notin S. \tag{3.21}$$

Substitution of Eq.(3.21) into Hooke's law leads to the following representation integral for the traction components at $\underline{x}$

$$f_p^{sc}(\underline{x}) = - C_{pq\ell k} n_q(\underline{x}) \int_S \{[u_{m,n}^{sc}(\underline{y})\sigma_{mn\ell}^G(\underline{x}-\underline{y}) - \rho\omega^2 u_i^{sc}(\underline{y})u_{i\ell}^G(\underline{x}-\underline{y})]\delta_{jk} -$$

$$- u_{i,k}^{sc}(\underline{y})\sigma_{ij\ell}^G(\underline{x}-\underline{y}) - \sigma_{ij}^{sc}(\underline{y})u_{i\ell,k}^G(\underline{x}-\underline{y})\} n_j dS(\underline{y}) \; , \; \underline{x} \notin S. \tag{3.22}$$

Equation (3.22) is a generalization of Hu's results for elastostatics (1987).

Application of Eq.(3.22) to a 3-D crack yields

$$f_p^{sc}(\underset{\sim}{x}) = - C_{pq\ell k} n_q(\underset{\sim}{x}) \int_{A^+} \{[\Delta u_{m,n}(\underset{\sim}{y}) \sigma^G_{mn\ell}(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 \Delta u_i(\underset{\sim}{y}) u^G_{i\ell}(\underset{\sim}{x}-\underset{\sim}{y})] \cdot \delta_{jk}$$

$$- \Delta u_{i,k}(\underset{\sim}{y}) \sigma^G_{ij\ell}(\underset{\sim}{x}-\underset{\sim}{y})\} n_j dA(\underset{\sim}{y}) \quad , \quad \underset{\sim}{x} \notin A^+. \tag{3.23}$$

where $\Delta u_i$ are the crack opening displacements and $\Delta u_{i,k}$ are their derivatives with respect to $y_k$. The last term of Eq.(3.22) disappears because of the continuity of $\sigma_{ij}n_j$ across the crack faces. BIE's are obtained by letting $\underset{\sim}{x} \to A^+$ as

$$f_p^{in}(\underset{\sim}{x}) = C_{pq\ell k} n_q(\underset{\sim}{x}) \int_{A^+} \{[\Delta u_{m,n}(\underset{\sim}{y}) \sigma^G_{mn\ell}(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 \Delta u_i(\underset{\sim}{y}) u^G_{i\ell}(\underset{\sim}{x}-\underset{\sim}{y})] \cdot \delta_{jk}$$

$$- \Delta u_{i,k}(\underset{\sim}{y}) \sigma^G_{ij\ell}(\underset{\sim}{x}-\underset{\sim}{y})\} n_j dA(\underset{\sim}{y}) \quad , \quad \underset{\sim}{x} \in A^+. \tag{3.24}$$

The integral of (3.24) is understood in the sense of Cauchy principal values. No extra discontinuity terms enter (3.24) as $\underset{\sim}{x} \to A^+$. Equation (3.24) is valid for a 3-D crack of arbitrary shape. The corresponding BIE's

for a 2-D crack in plane strain and anti-plane strain can be derived directly from (3.24). For plane strain we obtain

$$f_\alpha^{in}(\underset{\sim}{x}) = C_{\alpha\beta\gamma\epsilon} n_\beta(\underset{\sim}{x}) \int_{\Gamma^+} \{[\Delta u_{\xi,\eta}(\underset{\sim}{y})\sigma^g_{\xi\eta\gamma}(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 \Delta u_\delta(\underset{\sim}{y}) \cdot$$

$$u^g_{\delta\gamma}(\underset{\sim}{x}-\underset{\sim}{y})]\delta_{\lambda\epsilon} - \Delta u_{\mu,\epsilon}(\underset{\sim}{y})\sigma^g_{\mu\lambda\gamma}(\underset{\sim}{x}-\underset{\sim}{y})\} n_\lambda ds(\underset{\sim}{y}) \;, \quad \underset{\sim}{x} \in \Gamma^+, \tag{3.25}$$

while for anti-plane strain we find

$$f_3^{in}(\underset{\sim}{x}) = \mu \int_{\Gamma^+} \{[\Delta u_{3,\alpha}(\underset{\sim}{y})\sigma^g_{3\alpha 3}(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 \Delta u_3(\underset{\sim}{y}) u^G_{33}(\underset{\sim}{x}-\underset{\sim}{y})]\delta_{\beta\gamma}$$

$$- \Delta u_{3,\beta}(\underset{\sim}{y})\sigma^G_{3\gamma 3}(\underset{\sim}{x}-\underset{\sim}{y})\} n_\gamma ds(\underset{\sim}{y}) \;, \quad \underset{\sim}{x} \in \Gamma^+. \tag{3.26}$$

Here $\Gamma^+$ denotes the insonified side of the 2-D crack (see Fig. 3) and the superscript "g" represents the 2-D Green's functions (Appendix A). All integrals of (3.25) and (3.26) are understood as Cauchy principal values.

The BIE's (3.24) (as well as (3.25) and (3.26)) have the advantage over the conventional BIE formulation, i.e., Eq.(2.8), that no higher order singularities appear. The unknown boundary quantities in the new formulation are the crack opening displacements and their derivatives, where the latter have the physical meaning of dislocation densities. We note also that the procedure in deriving (3.24) is very natural, and no elaborate manipulations, such as integration by parts, have been used. The BIE's stated here apply also to elastostatic crack analysis by letting $\omega \to 0$, and by using the corresponding elastostatic fundamental solution (Kelvin solution). For the static case, the term containing $\Delta u_i$ disappears in the

BIE's, and the only unknown quantities are the dislocation densities. The new formulation allows an immediate numerical implementation. When $\Delta u_i$ has been computed equation (2.7) can be employed to calculate the displacement field.

As pointed out by Carlsson (1974), Bui (1977), and Moran and Shih (1987), there exists a complementary integral to $J_k$ which is also path independent. For time-harmonic elastodynamics, the complementary integral may be stated as

$$\tilde{J}_k^c = \int_S [(W^c + L^c)\delta_{jk} - \sigma_{ij,k}u_i]n_j dS = 0 \,, \tag{3.27}$$

where $W^c$ and $L^c$ are defined by the Legendre transformation

$$W^c = \sigma_{ij}\epsilon_{ij} - W \,, \tag{3.28}$$

$$L^c = \rho\ddot{u}_i u_i - L \,, \tag{3.29}$$

and W and L are given by (3.2) and (3.5). The assumptions in deriving (3.27) are the same as for $\tilde{J}_k$. The proof of Eq.(3.27) can be performed directly by using the divergence theorem, by considering the relation

$$\epsilon_{ij} = \frac{\partial W^c}{\partial \sigma_{ij}} \,, \tag{3.30}$$

and by employing Eq.(2.1).

Following the same procedure as in deriving Eq.(3.22), a novel representation integral for the scattered traction components $f_p^{sc}(\underset{\sim}{x})$ is obtained from (3.27) as

$$f_p^{sc}(\underset{\sim}{x}) = - C_{pq\ell k}n_q(\underset{\sim}{x}) \int_S \{[u_{m,n}^{sc}(\underset{\sim}{y})\sigma_{mn\ell}^G(\underset{\sim}{x}-\underset{\sim}{y}) - \rho\omega^2 u_i^{sc}(\underset{\sim}{y})u_{i\ell}^G(\underset{\sim}{x}-\underset{\sim}{y})]\delta_{jk}$$

$$- \sigma_{ij,k}^{sc}(\underset{\sim}{y})u_{i\ell}^G(\underset{\sim}{x}-\underset{\sim}{y}) - u_i^{sc}(\underset{\sim}{y})\sigma_{ij\ell,k}^G(\underset{\sim}{x}-\underset{\sim}{y})\}n_j dS(\underset{\sim}{y}) \ , \ \underset{\sim}{x} \notin S. \qquad (3.31)$$

Corresponding BIE's can be derived by applying Eq.(3.31) to the crack faces and by letting $\underset{\sim}{x} \to A$, taking into account of the boundary conditions (2.5). Such BIE's are, however, again highly singular due to the presence of the term $\sigma_{ij\ell,k}^G(\underset{\sim}{x}-\underset{\sim}{y})$. Hence, Eq.(3.31) has no advantages over the conventional formulation given by Eq.(2.8).

## 4. Examples

In this section we will apply the BIE's (3.24) (as well (3.25) and (3.26)), which are valid for arbitrary shaped cracks, to some simple cases. We first consider a flat 3-D crack in an unbounded body subjected to an incident time-harmonic wave. The crack is located in the plane $x_3 = 0^{\pm}$. Hence $n_1 = n_2 = 0$ and $n_3 = 1$. The BIE's (3.24) separate into two decoupled equations:

$$\sigma_{33}^{in}(x_1,x_2,0) = \int_{A^+} \{[(\lambda+2\mu)\sigma_{\alpha 33}^G(\underset{\sim}{x}-\underset{\sim}{y}) - \lambda\sigma_{33\alpha}^G(\underset{\sim}{x}-\underset{\sim}{y})]\Delta u_{3,\alpha}(\underset{\sim}{y}) -$$

$$- \rho\omega^2(\lambda+2\mu)u^G_{33}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_3(\underset{\sim}{y})\}dA(\underset{\sim}{y}),\ \underset{\sim}{x} \in A^+ \quad , \tag{4.1}$$

$$\sigma^{in}_{\beta 3}(x_1,x_2,0) = \mu\int_{A^+}\{[\sigma^G_{\alpha\gamma\beta}(\underset{\sim}{x}-\underset{\sim}{y}) - \sigma^G_{\alpha 33}(\underset{\sim}{x}-\underset{\sim}{y})\delta_{\beta\gamma}]\Delta u_{\alpha,\gamma}(\underset{\sim}{y}) -$$

$$\rho\omega^2 u^G_{\alpha\beta}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_\alpha(\underset{\sim}{y})\}dA(\underset{\sim}{y}),\ \underset{\sim}{x} \in A^+ \ ,\ \alpha,\beta = 1,2 \quad , \tag{4.2}$$

where $\sigma^{in}_{33}$ and $\sigma^{in}_{\beta 3}$ are stress components corresponding to the incident wave. We note that Eq.(4.1) is for the normal crack opening displacement $\Delta u_3$ while Eq.(4.2) is for the transverse crack opening displacements $\Delta u_\alpha$. Equations (4.1) and (4.2) have exactly the same forms as those derived by Budiansky and Rice (1979) who used the conventional formulation in conjunction with partial integration.

BIE's for a flat 3-D crack under static surface loading $\sigma_{33}(x_1,x_2,0)$ and $\sigma_{\beta 3}(x_1,x_2,0)$ can be obtained from (4.1) and (4.2) by letting $\omega \to 0$ and by employing the corresponding static fundamental solutions. The result is

$$\sigma_{33}(x_1,x_2,0) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+}\frac{r_{,\alpha}}{r^2}\,\Delta u_{3,\alpha}(\underset{\sim}{y})\ dA(\underset{\sim}{y}),\quad \underset{\sim}{x} \in A^+ \ , \tag{4.3}$$

$$\sigma_{\beta 3}(x_1,x_2,0) = \frac{\mu}{8\pi(1-\nu)}\int_{A^+}\frac{1}{r^2}\,\{(1-2\nu)[\delta_{\alpha\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta}] +$$

$$3r_{,\alpha}r_{,\beta}r_{,\gamma})\Delta u_{\alpha,\gamma}(\underset{\sim}{y})dA(\underset{\sim}{y}), \quad \underset{\sim}{x} \in A^{\pm} . \tag{4.4}$$

Here $r = |\underset{\sim}{x}-\underset{\sim}{y}|$, and $\nu$ denotes Poisson's ratio. Equations (4.3) and (4.4) are identical to the equations stated by Weaver (1977).

Next, we consider a straight 2-D crack for states of deformation in plane strain and anti-plane strain. The crack is defined by $x_2 = 0^{\pm}$, $|x_1| \leq a$. For the case of plane strain, the BIE's (3.25) become

$$\sigma^{in}_{12}(x_1,0) = \mu \int_{-a}^{a} \{[\sigma^g_{111}(\underset{\sim}{x}-\underset{\sim}{y}) - \sigma^g_{122}(\underset{\sim}{x}-\underset{\sim}{y})]\Delta u_{1,1}(y_1) -$$

$$-\rho\omega^2 u^g_{11}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_1(y_1)\}dy_1, \quad |x_1| \leq a , \tag{4.5}$$

$$\sigma^{in}_{22}(x_1,0) = \int_{-a}^{a} \{[(\lambda + 2\mu)\sigma^g_{112}(\underset{\sim}{x}-\underset{\sim}{y}) - \lambda\sigma^g_{221}(\underset{\sim}{x}-\underset{\sim}{y})]\Delta u_{2,1}(y_1) -$$

$$-\rho\omega^2(\lambda + 2\mu)u^g_{22}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_2(y_1)\}dy_1, \quad |x_1| \leq a , \tag{4.6}$$

while for anti-plane strain Eq.(3.26) takes the following form

$$\sigma^{in}_{32}(x_1,0) = \mu \int_{-a}^{a} \{[\sigma^g_{313}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_{3,1}(y_1) - \rho\omega^2 u^g_{33}(\underset{\sim}{x}-\underset{\sim}{y})\Delta u_3(y_1)\}dy_1,$$

$$|x_1| \leq a . \tag{4.7}$$

The BIE's (4.5) and (4.6) have been derived by Tan (1975) via the conventional formulation. For plane strain deformation, the BIE for the normal crack opening displacement, $\Delta u_2$ (Mode I), decouples from the one for the transverse crack opening displacement, $\Delta u_1$ (Mode II). For a 2-D crack under static loading $\sigma_{12}(x_1,0)$, $\sigma_{22}(x_1,0)$, and $\sigma_{32}(x_1,0)$ we obtain from (4.5) and (4.6)

$$\sigma_{12}(x_1,0) = \frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \frac{\Delta u_{1,1}}{x_1-y_1}\, dy_1 \ , \quad |x_1| \le a \ , \tag{4.8}$$

$$\sigma_{22}(x_1,0) = \frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \frac{\Delta u_{2,1}}{x_1-y_1}\, dy_1 \ , \quad |x_1| \le a \ , \tag{4.9}$$

for plane strain, and from (4.7)

$$\sigma_{32}(x_1,0) = \frac{1}{2\pi} \int_{-a}^{a} \frac{\Delta u_{3,1}}{x_1-y_1}\, dy_1 \ , \quad |x_1| \le a \ , \tag{4.10}$$

for anti-plane strain. Equations (4.8) - (4.10) are integral equations for dislocation densities, and they are again well known (see Mura, 1987).

The BIE's presented here must, in general, be solved numerically. Special care must be taken in the numerical implementation to account for the local behavior of $\Delta u_i$ and $\Delta u_{i,j}$ near crack edges, and for the singularities of the Green's functions at $\underset{\sim}{x} = \underset{\sim}{y}$. For 2-D cracks subjected by static loading the method developed by Erdogan et al. (1973) has been

frequently used, while Zhang and Achenbach (1988) solved the modified BIE's of (4.5) and (4.6) numerically for time-harmonic wave scattering problems. For a flat 3-D crack numerical methods have been proposed by Polch et al. (1987) for the static case, and by Nishimura and Kobayashi (1987) for the dynamic case.

## 5. Concluding Comments

A novel application of an elastodynamic conservation integral, the $\tilde{J}_k$ integral, to elastodynamic and elastostatic crack analysis has been presented. Boundary integral equations follow from $\tilde{J}_k$ in a direct and natural way. These equations immediately have lower order singularities than the ones obtained in the conventional manner by the use of the Betti-Rayleigh reciprocity integral. This is an important advantage for the development of a numerical procedure for solving the BIE's, and for an accurate calculation of the strains and stresses at internal points close to the crack faces. For 3-D or 2-D cracks of arbitrary shapes the BIE's presented here have simple forms, and they do not require integration by parts, as in the conventional formulation. In the dynamic case, the unknown quantities are the crack opening displacements and their derivatives (dislocation densities), while in the static case only the dislocation densities appear in the formulation. Thus, higher order shape functions for $\Delta u_i$ are desirable in the dynamic case. The complementary conservation integral $\tilde{J}_k^c$ gives rise to more singular BIE's which offer no advantages over the conventional equations.

The representation integral for the traction components, Eq.(3.23), can be used to derive BIE's for general boundary value problems (not necessary

cracks) of time-harmonic elastodynamics or elastostatics. The advantages and drawbacks of this approach compared to the conventional formulation have been discussed in the paper by Hu (1987).

As pointed out by Hu (1987), new BIE's can be derived from other conservation integrals. Following essentially the same procedure as described in section 3, the present authors have obtained another set of representation formulas for combinations of $u_i(\underset{\sim}{x})$ and $u_{i,j}(\underset{\sim}{x})$ from the well known M and $L_k$ integrals (see Knowles and Sternberg (1972), Budiansky and Rice (1973)). The significance of these representation formulas and their associated BIE's for solving boundary value problems is under investigation.

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## Appendix A: Green's functions

The Green's function for the 3-D time-harmonic elastodynamic state is given by (see Tan, 1975, Achenbach et al., 1982)

$$u^G_{ik}(\underset{\sim}{x}-\underset{\sim}{y}) = \frac{1}{4\pi\rho\omega^2}\left[\frac{\exp(ik_T r)}{r} - \frac{\exp(ik_L r)}{r}\right]_{,ik} + \frac{\exp(ik_T r)}{4\pi\mu r}\,\delta_{ik}\ , \qquad (A.1)$$

where

$$r = |\underset{\sim}{x} - \underset{\sim}{y}|\ . \qquad (A.2)$$

The function $u^G_{ik}(\underset{\sim}{x}-\underset{\sim}{y})$ denotes the displacement in the i-direction observed at position $\underset{\sim}{x}$ due a unit force in the k-direction, applied at position $\underset{\sim}{y}$. The corresponding components of the stress tensor follow from Hooke's law

$$\sigma^G_{ijk} = C_{ijmn}u^G_{mk,n}\ . \qquad (A.3)$$

Similarly, the Green's functions for the 2-D plane strain and anti-plane strain time-harmonic elastodynamic states are

$$u^g_{\alpha\gamma}(\underset{\sim}{x}-\underset{\sim}{y}) = \frac{i}{4\rho\omega^2}\left\{[H^{(1)}_o(k_T r) - H^{(1)}_o(k_L r)]_{,\alpha\gamma} + k_T^2\delta_{\alpha\gamma}H^{(1)}_o(k_T r)\right\}\ , \qquad (A.4)$$

$$u^g_{33}(\underset{\sim}{x}-\underset{\sim}{y}) = \frac{i}{4\mu}\,H^{(1)}_o(k_T r)\ , \qquad (A.5)$$

respectively, where $H_o^{(1)}(\cdot)$ denotes the Hankel function of the first kind and zeroth order. Expressions for the stress components can be obtained by using

$$\sigma^g_{\alpha\beta\gamma} = C_{\alpha\beta\xi\eta} u^g_{\xi\gamma,\eta} \quad , \quad \text{for plane strain} \quad , \tag{A.6}$$

$$\sigma^g_{3\alpha 3} = \mu \, u^g_{33,\alpha} \quad , \quad \text{for anti-plane strain} \quad . \tag{A.7}$$

The Kelvin solution for the 3-D elastostatics case may be written as

$$u^g_{ik}(\underset{\sim}{x}-\underset{\sim}{y}) = \frac{1}{16\pi\mu(1-\nu)r} \left[(3-4\nu)\delta_{ik} + r_{,i} r_{,k}\right] \quad , \tag{A.8}$$

while for 2-D elastostatics we have

$$u^g_{\alpha\gamma}(\underset{\sim}{x}-\underset{\sim}{y}) = -\frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu)\delta_{\alpha\gamma}\ell nr - r_{,\alpha} r_{,\gamma}\right] \quad , \tag{A.9}$$

$$u^g_{33}(\underset{\sim}{x}-\underset{\sim}{y}) = -\frac{1}{2\pi\mu} \ell nr \quad , \tag{A.10}$$

for plane strain and anti-plane strain, respectively. The corresponding stress components for elastostatics follow from the Eqs.(A.3), (A.6) and (A.7).

We note that both the dynamic and the static Green's functions possess the same singularities at r = 0, namely,

$$\left.\begin{aligned} u^G_{ik} &\sim \frac{1}{r} \\ \sigma^G_{ijk} &\sim \frac{1}{r^2} \\ \sigma^G_{ijk,\ell} &\sim \frac{1}{r^3} \end{aligned}\right\} \quad \text{as } r \to 0 \quad , \tag{A.11}$$

for the 3-D case, and

$$\left.\begin{aligned} u^g_{\alpha\beta} \; , \quad u^g_{33} &\sim \ell nr \\ \sigma^g_{\alpha\beta\gamma} \; , \quad \sigma^g_{3\alpha 3} &\sim \frac{1}{r} \\ \sigma^g_{\alpha\beta\gamma,\delta} \; , \quad \sigma^g_{3\alpha 3,\beta} &\sim \frac{1}{r^2} \end{aligned}\right\} \quad \text{as } r \to 0 \quad , \tag{A.12}$$

for 2-D plane strain and anti-plane strain. All derivatives in the Green's functions are understood to be with respect to $\underset{\sim}{y}$.

## Appendix B: Asymptotic Expansions of the Elastodynamic Green's Functions

For $|\underset{\sim}{y}| >> |\underset{\sim}{x}|$, the following approximation holds

$$r = |\underset{\sim}{x}-\underset{\sim}{y}| \simeq |\underset{\sim}{y}| - \hat{\underset{\sim}{y}}\cdot\underset{\sim}{x} \quad , \tag{B.1}$$

where $\hat{\underset{\sim}{y}}$ denotes the unit vector along $\underset{\sim}{y}$. By using (B.1), asymptotic expressions for the 3-D elastodynamic Green's functions are obtained as

$$u^G_{i\ell}(\underset{\sim}{x}-\underset{\sim}{y}) \simeq \sum_{\xi=L,T} A^{\xi}_{i\ell}(\hat{\underset{\sim}{y}}) \frac{\exp(ik_\xi|\underset{\sim}{y}|)}{4\pi|\underset{\sim}{y}|} \exp(-ik_\xi\hat{\underset{\sim}{y}}\cdot\underset{\sim}{x}) \quad , \tag{B.2}$$

$$u^G_{i\ell,k}(\underset{\sim}{x}-\underset{\sim}{y}) \simeq \sum_{\xi=L,T} ik_\xi \, B^\xi_{i\ell k}(\hat{\underset{\sim}{y}}) \frac{\exp(ik_\xi|\underset{\sim}{y}|)}{4\pi|\underset{\sim}{y}|} \exp(-ik_\xi \hat{\underset{\sim}{y}}\cdot\underset{\sim}{x}) \quad , \tag{B.3}$$

$$\sigma^G_{ijk}(\underset{\sim}{x}-\underset{\sim}{y}) \simeq \sum_{\xi=L,T} ik_\xi \, C^\xi_{ijk}(\hat{\underset{\sim}{y}}) \frac{\exp(ik_\xi|\underset{\sim}{y}|)}{4\pi|\underset{\sim}{y}|} \exp(-ik_\xi \hat{\underset{\sim}{y}}\cdot\underset{\sim}{x}) \quad , \tag{B.4}$$

in which

$$A^L_{i\ell}(\hat{\underset{\sim}{y}}) = \hat{y}_i\hat{y}_\ell/(\lambda+2\mu) \quad , \tag{B.5}$$

$$A^T_{i\ell}(\hat{\underset{\sim}{y}}) = (\delta_{i\ell} - \hat{y}_i\hat{y}_\ell)/\mu \quad , \tag{B.6}$$

$$B^L_{i\ell k}(\hat{\underset{\sim}{y}}) = \hat{y}_i\hat{y}_\ell\hat{y}_k/(\lambda+2\mu) \quad , \tag{B.7}$$

$$B^T_{i\ell k}(\hat{\underset{\sim}{y}}) = (\delta_{i\ell} - \hat{y}_i\hat{y}_\ell)\hat{y}_k/\mu \quad , \tag{B.8}$$

$$C^L_{ijk}(\hat{\underset{\sim}{y}}) = [2\kappa^{-2}\hat{y}_i\hat{y}_j + (1-2\kappa^{-2})\delta_{ij}]\hat{y}_k \quad , \tag{B.9}$$

$$C^T_{ijk}(\hat{\underset{\sim}{y}}) = \delta_{ik}\hat{y}_j + \delta_{jk}\hat{y}_i - 2\hat{y}_i\hat{y}_j\hat{y}_k, \tag{B.10}$$

$$\kappa = k_T/k_L \quad . \tag{B.11}$$

Also, $k_L$ and $k_T$ are the wavenumbers of longitudinal and transverse waves, respectively.

For a large sphere of radius R (see Fig. 2), the following relations hold

$$|\underset{\sim}{y}| \simeq R \quad , \tag{B.12}$$

$$r_{,k} = \hat{y}_k = n_k \quad , \tag{B.13}$$

where $n_k$ is the components of the unit outward normal vector of the sphere. Substitution of Eqs.(B.2) - (B.13) into Eqs.(3.13) and (3.14) yields Eq.(3.16).

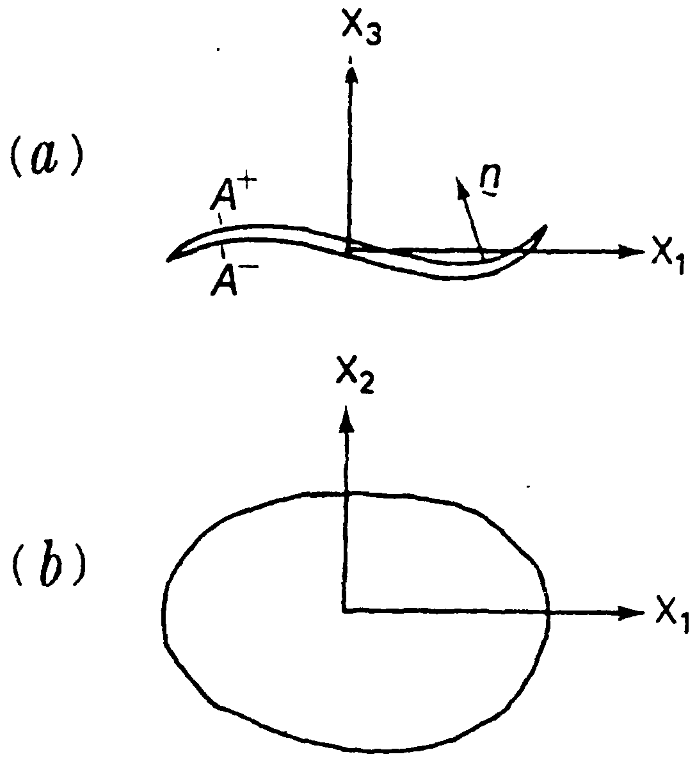


Fig. 1: Curved crack of arbitrary shape;
(a) $x_1x_3$-plane, (b) top view.

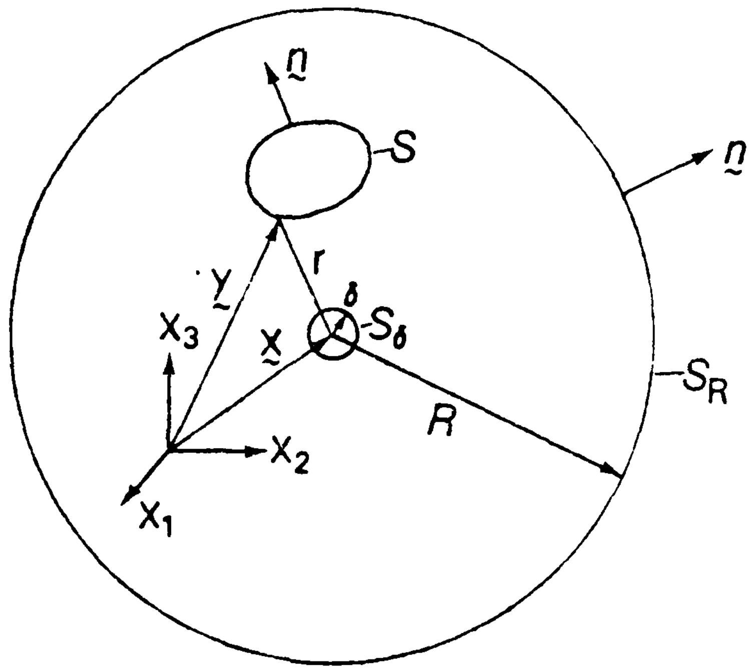


Fig. 2: A scatterer in an unbounded solid.

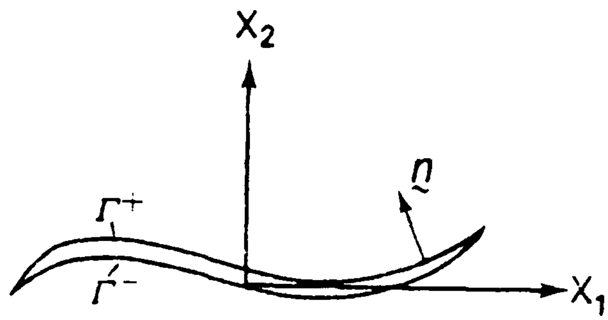


Fig. 3: A curved crack in a 2-D geometry.

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An elastodynamic conservation integral, the $J_k$ integral, is employed to derive boundary integral equations for crack configurations, in a direct and natural way. These equations immediately have lower order singularities than the ones obtained in the conventional manner by the use of the Betti-Rayleigh reciprocity relation. This is an important advantage for the development of numerical procedures for solving the BIE's, and for an accurate calculation of the strains and stresses at internal points close to the crack faces. For curved cracks of arbitrary shape the BIE's presented here have simple forms,

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and they do not require integration by parts, as in the conventional formulation. For the dynamic case, the unknown quantities are the crack opening displacements and their derivatives (dislocation densities), while for the static case only the dislocation densities appear in the formulation. For plane cracks the boundary integral equations reduce to the ones obtained by other investigators.

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